THE SCHWARZIAN DERIVATIVE FOR HARMONIC MAPPINGS

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§1. Introduction.

The Schwarzian derivative of an analytic function is a basic tool in complex analysis. It appeared as early as 1873, when H. A. Schwarz sought to generalize the Schwarz–Christoffel formula to conformal mappings of polygons bounded by circular arcs. More recently, Nehari [5, 6, 7] and others have developed important criteria for global univalence in terms of the Schwarzian derivative, exploiting its connection with linear differential equations. Osgood and Stowe [8] have unified these various univalence criteria through a general theorem involving the curvature of a metric.

The purpose of the present paper is to offer a definition of Schwarzian derivative that applies more generally to complex-valued *harmonic* functions. The formula is derived in a natural way by passing to the minimal surface associated locally with a given harmonic function. The derivation then appeals to a definition given by Osgood and Stowe [9] for the Schwarzian derivative of a conformal mapping between arbitrary Riemannian manifolds. The resulting expression reduces to standard form when the harmonic function is analytic, and various classical properties of Schwarzian derivatives generalize in appropriate ways, suggesting that the definition we propose is the "right" one.

The Schwarzian derivative of a locally univalent analytic function f is defined by

$$S(f) = (f''/f')' - \tfrac{1}{2}(f''/f')^2 \,.$$

The key property is its invariance under postcomposition with Möbius transformations. If

$$T(z) = \frac{az+b}{cz+d}\,, \qquad ad - bc \neq 0,$$

is any Möbius transformation, then $S(T \circ f) = S(f)$. This is a special case of the transformation formula

$$S(g \circ f) = (S(g) \circ f)(f')^2 + S(f)\,,$$

since $S(T) = 0$ for Möbius transformations. Note also that $S(f \circ T) = (S(f) \circ T)(T')^2$.

For an arbitrary analytic function φ, the general function f with Schwarzian $S(f) = 2\varphi$ has the form $f = w_1/w_2$, where w_1 and w_2 are arbitrary linearly independent solutions of the differential equation $w'' + \varphi w = 0$. Two consequences are:

(*i*) If $S(f) = 0$, then f is a Möbius transformation;

(*ii*) If $S(g) = S(f)$, then $g = T \circ f$ for some Möbius transformation T.

To set the stage for later discussion, we now give a brief overview of harmonic mappings and their associated minimal surfaces. Details and further information may be found in [2] or [10]. We use the standard notation

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\,\frac{\partial}{\partial y}\right), \qquad \frac{\partial}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\,\frac{\partial}{\partial y}\right)$$

for the z- and $\overline{z}$-derivatives, where $z = x + iy$. Recall that the Laplacian is given by

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = 4\,\frac{\partial^2}{\partial z \partial \overline{z}}\,.$$

A complex-valued function f that is harmonic in a simply connected domain $\Omega \subset \mathbb{C}$ has the canonical representation $f = h + \overline{g}$, where f and g are analytic in Ω and $g(z_0) = 0$ for some prescribed point $z_0 \in \Omega$. According to a theorem of H. Lewy [4], f is locally univalent if and *only* if its Jacobian $|f_z|^2 - |f_{\overline{z}}|^2 = |h'|^2 - |g'|^2$ does not vanish. It is said to be *sense-preserving* if its Jacobian is positive. Then $h'(z) \neq 0$ and the analytic function $\omega = g'/h'$, called the *(second) complex dilatation* of f, has the property $|\omega(z)| < 1$ in Ω. Throughout this paper we will assume that f is locally univalent and sense-preserving, and we call f a *harmonic mapping*.

The harmonic mappings with dilatation $\omega(z) \equiv 0$ are precisely the conformal mappings. More generally, it is easily seen that harmonic mappings with constant dilatation $\omega(z) \equiv \alpha$ have the form $f = h + \overline{\alpha h}$ for some analytic locally univalent function h.

A harmonic mapping $f = h + \overline{g}$ can be lifted locally to a regular minimal surface given by conformal (or isothermal) parameters if and only if its dilatation is the square of an analytic function: $\omega = q^2$ for some analytic function q with $|q(z)| < 1$. Equivalently, the requirement is that any zero of ω be of even order, unless $\omega(z) \equiv 0$. For such harmonic mappings $f = u + iv$, the minimal surface has the *Weierstrass–Enneper representation*

$$\begin{aligned} u &= \mathrm{Re}\left\{ \int_{z_0}^{z} p(1+q^2)\,d\zeta \right\}, \\ v &= \mathrm{Im}\left\{ \int_{z_0}^{z} p(1-q^2)\,d\zeta \right\}, \\ w &= 2\,\mathrm{Im}\left\{ \int_{z_0}^{z} pq\,d\zeta \right\}, \end{aligned}$$

where $p = h'$ and $w = F(u, v)$ is the height of the surface. The metric of the surface has the form $ds = \rho|dz|$, where $\rho = \rho(z) > 0$. Here the function ρ takes the form

$$\rho = |h'| + |g'| = |h'|(1 + |\omega|) = |p|(1 + |q|^2).$$

A general theorem of differential geometry says that if any regular surface is represented by conformal parameters, so that its metric has the form $ds = \rho|dz|$ for some for some positive function ρ, then the Gauss curvature of the surface is $K = -\rho^{-2}\,\Delta(\log\rho)$, where Δ denotes the Laplacian. This quantity K is also known as the curvature of the metric. In our special case of a minimal surface associated with a harmonic mapping $f = h + \overline{g}$, the formula for curvature reduces to

$$K = -\frac{4|q'|^2}{|p|^2(1 + |q|^2)^4},$$

where $p = h'$ and $q^2 = \omega = g'/h'$.

Our definition of the Schwarzian derivative of a harmonic mapping f is given in terms of the metric $ds = \rho|dz|$ of the associated minimal surface. The formula is

$$S(f) = 2\big\{(\log\rho)_{zz} - ((\log\rho)_z)^2\big\}.$$

The geometric considerations that lead to this formula will be described in Section 4. First, however, we shall take the formula as given and explore some of its basic properties.

§2. Properties of the generalized Schwarzian.

Observe first that if f is *analytic*, then $\rho = |f'|$ and $\omega = 0$, so that the associated minimal surface is the uv–plane. Since

$$\log\rho = \tfrac{1}{2}(\log f' + \log\overline{f'}),$$

we see that $(\log\rho)_z = \frac{1}{2}f''/f'$, and the generalized Schwarzian takes the form

$$S(f) = 2\big\{(\log\rho)_{zz} - ((\log\rho)_z)^2\big\} = (f''/f')' - \tfrac{1}{2}(f''/f')^2,$$

in agreement with the classical formula for the Schwarzian derivative of an analytic function. More generally, if $f = h + \alpha\overline{h}$, where h is analytic and α is a complex constant with $|\alpha| \neq 1$, then $\rho = (1 + |\alpha|)|h'|$ and again $S(f) = S(h)$.

If $f = h + \overline{g}$ is an arbitrary harmonic mapping with dilatation $\omega = q^2$, then we can write

$$\rho = |h'|(1 + |q|^2) = |h'|(1 + q\overline{q}),$$

which gives

$$(\log \rho)_z = \tfrac{1}{2}\,\frac{h''}{h'} + \frac{q'\,\overline{q}}{1+|q|^2}\,.$$

Further calculations then lead to the expression

$$S(f) = S(h) + \frac{2\,\overline{q}}{1+|q|^2}\left(q'' - \frac{q'h''}{h'}\right) - 4\left(\frac{q'\,\overline{q}}{1+|q|^2}\right)^2.$$

Note again that $S(f) = S(h)$ when $\omega = q^2$ is constant. It should also be observed that the Schwarzian derivative is well defined (and single-valued) in a deleted neighborhood of a point where ω has a zero of odd order, since the above formula is invariant under change of sign in $q = \sqrt{\omega}$. For instance, if f has dilatation $\omega(z) = z$, the formula reduces to

$$S(f) = S(h) - \frac{|z|}{(1+|z|)z}\left(\frac{1}{2z} + \frac{h''}{h'}\right) - \left(\frac{|z|}{(1+|z|)z}\right)^2.$$

If q has a zero of order at least 2 at some point z_0, then $S(f)$ tends to $S(h)$ as $z \to z_0$.

If $f = h + \overline{g}$ is sense-reversing, its Schwarzian derivative can be defined in a similar manner. Because of the symmetry in the formula $\rho = |h'| + |g'|$, it is clear that $S(\overline{f}) = S(f)$. Thus there is no essential loss of generality in restricting attention to Schwarzians of *sense-preserving* harmonic mappings.

We now investigate the behavior of the generalized Schwarzian under composition. If φ is a locally univalent analytic function for which the composition $f \circ \varphi$ is defined, then $f \circ \varphi$ is again a harmonic mapping and

$$\rho_{f\circ\varphi} = (\rho_f \circ \varphi)|\varphi'|\,.$$

A calculation then gives

$$S(f\circ\varphi) = (S(f)\circ\varphi)(\varphi')^2 + S(\varphi)\,,$$

a direct generalization of the transformation formula for the classical Schwarzian of an analytic function f.

As for postcompositions, if f is harmonic and L is an affine mapping of $\mathbb{R}^2$, then $L \circ f$ is harmonic. However, the Schwarzian of $L \circ f$ *will not be equal* to that of f unless L is conformal. This will be clear from results in the next section. When L is a conformal affine map, then ρ_f and $\rho_{L\circ f}$ are proportional by a constant factor, hence the Schwarzians are equal.

We next show that the Schwarzian $S(f)$ is analytic only for harmonic mappings of the form $f = h + \alpha\overline{h}$ with $|\alpha| < 1$. In fact, we shall prove the following theorem.

Theorem 1. *For a harmonic mapping f with dilatation $\omega = q^2$, the following are equivalent:*

(*i*) *$S(f)$ is analytic.*
(*ii*) *The curvature K of the conformal metric associated with f is constant.*
(*iii*) *$K \equiv 0$, so that the corresponding minimal surface is a plane.*
(*iv*) *The dilatation of f is constant.*
(*v*) *$f = h + \alpha\overline{h}$ for some analytic locally univalent function h and for some complex constant α with $|\alpha| < 1$.*
(*vi*) *The Weierstrass–Enneper lifting $\widetilde{f} = (u, v, w)$ of f to its corresponding minimal surface has the form*

$$\tilde{f}(z) = A \begin{pmatrix} \mathrm{Re}\{h(z)\} \\ \mathrm{Im}\{h(z)\} \end{pmatrix} ,$$

where $A : \mathbb{R}^2 \to \mathbb{R}^3$ is a linear conformal mapping.

Proof. (*i*) $\Longrightarrow$ (*ii*). The curvature of the metric $\rho = \rho_f$ associated with f is

$$K = -\frac{1}{\rho^2}\,\Delta(\log\rho) = -\frac{4\,(\log\rho)_{z\overline{z}}}{\rho^2}\,.$$

A simple calculation yields

$$-\frac{1}{4}K_z = \frac{1}{\rho^2}\left[(\log\rho)_{zz} - ((\log\rho)_z)^2\right]_{\overline{z}} = \frac{1}{2\rho^2}[S(f)]_{\overline{z}} = 0$$

if $S(f)$ is analytic. Thus K is constant.

(*ii*) $\Longrightarrow$ (*iii*). This says that a minimal surface with constant curvature must lie in a plane. Referring to our formula for the curvature K in terms of the Weierstrass–Enneper parameters p and q, and passing to logarithms, we see that if K is constant, then

$$\log(1 + |q|^2) = \tfrac{1}{2}\,\log|q'/p| + c$$

for some constant c. Thus $\log(1 + |q|^2)$ is a harmonic function. But a calculation gives

$$\left[\log(1 + |q|^2)\right]_{z\overline{z}} = \left[\frac{q'\,\overline{q}}{1 + |q|^2}\right]_{\overline{z}} = \frac{|q'|^2}{(1 + |q|^2)^2}\,,$$

so $\log(1+|q|^2)$ is harmonic if and only if $q' = 0$. But then the formula for curvature shows that $K = 0$.

(*iii*) $\Longrightarrow$ (*iv*). If $K = 0$, then $q' = 0$, so q is constant and the dilatation $\omega = q^2$ is constant.

(*iv*) $\Longrightarrow$ (*v*). This was already noted above, but here are the details. If a harmonic mapping $f = h + \overline{g}$ has constant dilatation, then $g' = \alpha h'$ for some constant α with $|\alpha| < 1$. Integration gives $g = \alpha h + \beta$ for some constant β. But since $|\alpha| \neq 1$, a bit of linear algebra shows that the additive constant β can be absorbed into h and we can write, with slight change of notation, $f = h + \alpha\overline{h}$ for some analytic function h. Finally, since f is locally univalent and the affine mapping $z \mapsto z + \alpha\overline{z}$ is invertible, we see that h is locally univalent.

$(v) \implies (i)$. As observed above, $S(h+\alpha\overline{h}) = S(h)$. We now see that (i) through (v) are equivalent.

$(iv) \implies (vi) \implies (i)$. If q is a constant q_0, then one sees directly from the Weierstrass–Enneper representation, where $p = h'$, that

$$u = \mathrm{Re}\{(1+q_0^2)h(z)\}\,, \qquad v = \mathrm{Im}\{(1-q_0^2)h(z)\}\,, \qquad w = 2\,\mathrm{Im}\{q_0 h(z)\}\,,$$

taking $h(z_0) = 0$. Hence $\tilde{f}(z) = A\begin{pmatrix} \mathrm{Re}\{h(z)\} \\ \mathrm{Im}\{h(z)\} \end{pmatrix}$, where

$$A = \begin{pmatrix} \mathrm{Re}\{1+q_0^2\} & -\mathrm{Im}\{q_0^2\} \\ -\mathrm{Im}\{q_0^2\} & \mathrm{Re}\{1-q_0^2\} \\ 2\,\mathrm{Im}\{q_0\} & 2\,\mathrm{Re}\{q_0\} \end{pmatrix}.$$

The columns are orthogonal and both have length $1+|q_0|^2$. Thus (iv) implies (vi). Finally, if (vi) holds, then the metric $\rho|dz|$ induced by f is a multiple of $|h'(z)||dz|$, and $S(f)$ is analytic. Thus (vi) implies (i), and the proof is complete.

Remark. The proof of the theorem contains an observation about general conformal metrics, namely that $\rho|dz|$ has constant curvature if and only if $(\log\rho)_{zz} - ((\log\rho)_z)^2$ is an analytic function.

Recall now that the *analytic* functions with vanishing Schwarzian derivatives are precisely the Möbius transformations. With appeal to Theorem 1, we can now obtain a corresponding result for harmonic mappings.

Theorem 2. *A harmonic mapping f has vanishing Schwarzian derivative $S(f) = 0$ if and only if it has the form $f = h + \alpha\overline{h}$ for some Möbius transformation h and some complex constant α with $|\alpha| < 1$.*

Proof. If $f = h + \alpha\overline{h}$ for a Möbius transformation h, then $S(f) = S(h) = 0$. Conversely, suppose that a harmonic mapping $f = h + \overline{g}$ has Schwarzian derivative $S(f) = 0$. Then by Theorem 1, we can conclude that f has constant dilatation $\omega = g'/h'$, so that $|g'| = c|h'|$ for some constant $c \geq 0$. It follows that

$$\rho = |h'| + |g'| = (1+c)|h'|\,,$$

so that $0 = S(f) = S(h)$, and h is a Möbius transformation. Also, since $\omega = \alpha$ for some constant α with $|\alpha| < 1$, we see that $g = \alpha h + \beta$ for some constant β. Again the additive constant β can be absorbed into the Möbius transformation h, and so with change of notation we can write $f = h + \alpha\overline{h}$ for some Möbius transformation h.

The theorem shows that a harmonic mapping with $S(f) = 0$ is injective and extends to a harmonic mapping of $\mathbb{C}$ onto itself. We define a *harmonic Möbius transformation* to be a harmonic mapping of the form $f = h + \alpha\overline{h}$, where h is a (classical) Möbius transformation and α is a complex constant with $|\alpha| < 1$. Theorem 2 says that these are precisely the harmonic mappings with $S(f) = 0$. Since a harmonic Möbius transformation is the composition of a Möbius transformation with an affine mapping, we can see that a harmonic Möbius transformation maps circles to ellipses. The basic transformation formula shows that $S(f \circ \varphi) = S(\varphi)$ if φ is analytic and f is a harmonic Möbius transformation.

The next problem is to characterize the class of harmonic mappings with prescribed Schwarzian. This more difficult task will be carried out in the next two sections. We close the present section with some simple examples.

First consider the harmonic mapping $f(z) = z + \frac{1}{3}\overline{z}^3$, which has dilatation $\omega(z) = z^2$ and maps the unit disk onto the domain inside a hypocycloid of 4 cusps inscribed in the circle $|w| = \frac{4}{3}$. It lifts to the classical Enneper surface. Note that $\rho = |h'| + |g'| = 1 + |z|^2$, so that the Schwarzian derivative is

$$S(f) = -\frac{4\,\overline{z}^2}{(1 + |z|^2)^2}\,.$$

The canonical harmonic mapping f onto a square domain inscribed in the unit circle has dilatation $\omega(z) = z^2$ and lifts to Scherk's first surface. (See [2], Chapter 10.) It has the form $f = h + \overline{g}$, where

$$h'(z) = \frac{2\sqrt{2}}{\pi(1 + z^4)}\,, \qquad g'(z) = z^2 h'(z)\,.$$

Thus

$$\rho = |h'| + |g'| = \frac{2\sqrt{2}}{\pi}\,\frac{1 + |z|^2}{|1 + z^4|}\,,$$

and a calculation gives

$$S(f) = -\frac{12z^2}{(1 + z^4)^2} - 4\left(\frac{z^3}{1 + z^4} - \frac{\overline{z}}{1 + |z|^2}\right)^2.$$

Similarly, it can be shown [3] that the univalent harmonic mapping

$$f(z) = -\tfrac{1}{2}\sum_{n=1}^{4} i^n \log|z - i^n|$$

has dilatation $\omega(z) = z^2$ and lifts to Scherk's "saddle-tower" surface. It has the form $f = h + \overline{g}$, with

$$h'(z) = \frac{1}{1 - z^4}\,, \qquad g'(z) = \frac{z^2}{1 - z^4}\,.$$

Thus

$$\rho = |h'| + |g'| = \frac{1+|z|^2}{|1-z^4|}\,,$$

and its Schwarzian derivative is found to be

$$S(f) = -\frac{12z^2}{(1-z^4)^2} - 4\left(\frac{z^3}{1-z^4} - \frac{\overline{z}}{1+|z|^2}\right)^2.$$

Next consider the harmonic mapping

$$f(z) = \log\left|\frac{1+z}{1-z}\right| - \overline{z}\,,$$

which results from horizontal shearing of the conformal mapping $\varphi(z) = z$ with dilatation $\omega(z) = z^2$. Its range is symmetric with respect to the real and imaginary axes and it occupies an unbounded portion of the horizontal strip $|\mathrm{Im}\{w\}| < 1$. Here $f = h + \overline{g}$, where

$$h(z) = \tfrac{1}{2}\,\log\frac{1+z}{1-z}\,, \qquad g(z) = h(z) - z\,.$$

Thus

$$\rho = |h'| + |g'| = \frac{1+|z|^2}{|1-z^2|}\,,$$

and a simple calculation gives the Schwarzian derivative

$$S(f) = 2\left(\frac{1}{(1-z^2)^2} - \frac{2\overline{z}^2}{(1+|z|^2)^2} - \frac{2|z|^2}{(1+|z|^2)(1-z^2)}\right).$$

The general harmonic shear of a conformal mapping φ convex in the horizontal direction, with dilatation $\omega = q^2$, has the form $f = h + \overline{g}$, where $h - g = \varphi$ and $g' = q^2 h'$. (See [1] or [2].) Solving the pair of linear equations, one finds $h' = \varphi'/(1-q^2)$. A calculation then yields the formula

$$\begin{aligned} S(f) = S(\varphi) &+ \frac{2({q'}^2 + (1-q^2)qq'')}{(1-q^2)^2} - \frac{2qq'}{1-q^2}\frac{\varphi''}{\varphi'} \\ &+ \frac{2\,\overline{q}}{1+|q|^2}\left\{q'' - q'\left(\frac{\varphi''}{\varphi'} + \frac{2qq'}{1-q^2}\right)\right\} - 4\left(\frac{q'\,\overline{q}}{1+|q|^2}\right)^2. \end{aligned}$$

If φ is the Koebe function $k(z) = z/(1-z)^2$ and $q(z) = z$, the formula reduces to

$$S(f) = -4\left(\frac{1}{(1-z)^2} + \frac{\overline{z}}{1+|z|^2}\right)^2.$$

§3. Does $S(f)$ determine f?

Let us ask the question in another way. If two harmonic mappings f and F have the same Schwarzian derivative, how are f and F related? One form of the answer is given by the following theorem, where the curvatures of the associated conformal metrics play an essential role.

Theorem 3. *Let $f = h + \overline{g}$ and $F = H + \overline{G}$ be harmonic mappings of a common domain Ω. If $S(f) = S(F)$, then*

(a) *The curvatures of the associated conformal metrics are equal: $K(\rho_f) = K(\rho_F)$.*

(b) *If the curvatures are not constant, then the metrics are homothetic; that is, $\rho_f = c\,\rho_F$ for some constant $c > 0$.*

(c) *If the curvatures are constant, then both are zero, and $f = h + \alpha\overline{h}$, $F = H + \beta\overline{H}$, and $H = T(h)$ for some analytic univalent functions h and H, some complex constants α and β with $|\alpha| < 1$ and $|\beta| < 1$, and some analytic Möbius transformation T.*

Conversely, if either (b) or (c) holds, then the curvatures are equal and $S(f) = S(F)$.

The proof requires some preparation and will be given in Section 5. Here we shall examine some of the algebraic and geometric aspects of the condition that $\rho_f = c\,\rho_F$. In terms of the respective Weierstrass–Enneper parameters (p, q) and (P, Q), the equation is $|p|(1 + |q|^2) = c|P|(1 + |Q|^2)$. For the calculations that follow it will be a little easier to absorb the constant into P and thus to consider the condition as

$$|p|(1 + |q|^2) = |P|(1 + |Q|^2)\,. \tag{1}$$

How are the parameters (p, q) and (P, Q) related? One obvious possibility is that $P = e^{i\theta}p$ and $Q = e^{i\phi}q$. But there are other, less obvious possibilities.

We introduce the following transformations of the pair (p, q):

$$R_p(\theta): \quad (p, q) \mapsto (e^{i\theta}p, q)\,,$$

$$R_q(\phi): \quad (p, q) \mapsto (p, e^{i\phi}q)\,,$$

$$I(\xi): \quad (p, q) \mapsto \left(\frac{(q-\xi)^2}{1+|\xi|^2}\,p\,,\ \frac{\xi}{\overline{\xi}}\,\frac{\overline{\xi}q+1}{\xi - q}\right), \qquad \xi \in \mathbb{C}\,,\ \xi \neq 0\,,$$

$$I(0): \quad (p, q) \mapsto (pq^2, -1/q)\,.$$

The following relationships are readily verified:

$$R_p(\theta_1)R_p(\theta_2) = R_p(\theta_1{+}\theta_2)\,; \quad R_q(\phi_1)R_q(\phi_2) = R_q(\phi_1{+}\phi_2)\,; \quad R_p(0) = R_q(0) = \mathrm{Id}\,;$$

$$R_p(\theta)R_q(\phi) = R_q(\phi)R_p(\theta)\,; \quad R_p(\theta)I(\xi) = I(\xi)R_p(\theta)\,;$$

$$R_q(\phi)I(\xi)R_q(-\phi) = I(e^{i\phi}\xi)R_p(-2\phi)\,; \qquad I(\xi_1)I(\xi_2) = I(\xi_3)R_q(\phi)R_p(\theta) \qquad \text{for}$$

$$\xi_1 + \xi_2 \neq 0\,, \quad \xi_3 = \frac{\xi_1}{\overline{\xi_1}}\,\frac{\overline{\xi_1}\xi_2 - 1}{\xi_1 + \xi_2}\,, \quad e^{i\phi} = \frac{\xi_1}{\overline{\xi_1}}\,\frac{\overline{\xi_2}}{\xi_2}\,\frac{\overline{\xi_1}\xi_2 - 1}{\xi_1\overline{\xi_2} - 1}\,, \quad e^{i\theta} = e^{2i\phi}\,\frac{(\xi_1 + \xi_2)^2}{|\xi_1 + \xi_2|^2}\,;$$

$$I(\xi)I(-\xi) = R_p(\theta) \quad \text{for } e^{i\theta} = \frac{\xi^2}{\overline{\xi}^2}\,, \ \ \xi \neq 0\,; \quad I(0)^2 = \mathrm{Id}\,.$$

With these formulas available, it is clear that the transformations $R_p(\theta)$, $R_q(\phi)$, and $I(\xi)$ for varying θ, ϕ, and ξ generate a group $\mathcal{G}$ under composition. We will explain the geometry of these transformations in terms of the minimal surface, but first we intend to show that the pairs (p, q) and (P, Q) satisfy (1) if and only if they are related by an element of $\mathcal{G}$.

It is easy to see that if $(P, Q) = \psi(p, q)$ for some $\psi \in \mathcal{G}$ then the relation (1) holds. For this we need only check that the quantity $|p|(1 + |q|^2)$ is invariant under each of the transformations $R_p(\theta)$, $R_q(\phi)$, and $I(\xi)$. Suppose conversely that the pairs of functions (p, q) and (P, Q) satisfy equation (1). Our analysis will be local, so we may assume that P, Q, and p have no zeros. We may then write

$$1 + |q|^2 = \frac{|P|}{|p|}(1 + |Q|^2) = |m|^2 + |n|^2 \,, \tag{2}$$

where $m^2 = P/p$ and $n = mQ$. Take the Laplacian of both sides of (2) to get

$$|q'|^2 = |m'|^2 + |n'|^2 \,.$$

Unless m is constant, a case to be treated later, this can be written as

$$\frac{|q'|^2}{|m'|^2} = 1 + \frac{|n'|^2}{|m'|^2} \,.$$

The logarithm of the left-hand side is harmonic, while that of the right-hand side will be harmonic if and only if $n'/m' = \alpha$, a constant. Thus

$$n' = \alpha m' \,, \quad q' = e^{i\theta}\sqrt{1 + |\alpha|^2}\, m' \,,$$

and so for constants β, γ

$$n = \alpha m + \beta \,, \quad q = e^{i\theta}\sqrt{1 + |\alpha|^2}\, m + \gamma \,. \tag{3}$$

Inserting this into (2), we obtain

$$1+(1+|\alpha|^2)|m|^2+|\gamma|^2+2\,\mathrm{Re}\{e^{i\theta}\overline{\gamma}\sqrt{1+|\alpha|^2}\, m\} = |m|^2+|\alpha|^2|m|^2+|\beta|^2+2\,\mathrm{Re}\{\alpha\overline{\beta}m\} \,.$$

Hence

$$2\,\mathrm{Re}\{(e^{i\theta}\overline{\gamma}\sqrt{1+|\alpha|^2} - \alpha\overline{\beta})m\} = |\beta|^2 - |\gamma|^2 - 1 \,.$$

This means that unless $e^{i\theta}\overline{\gamma}\sqrt{1+|\alpha|^2} - \alpha\overline{\beta} = 0$, the values of m lie on a line. But this would make m constant, so we conclude that indeed

$$e^{i\theta}\overline{\gamma}\sqrt{1+|\alpha|^2} = \alpha\overline{\beta} \,, \tag{4}$$

and also that

$$|\beta|^2 = 1 + |\gamma|^2\,. \tag{5}$$

Taking absolute values in (4) gives $|\gamma|^2(1+|\alpha|^2) = |\alpha|^2|\beta|^2$, while from (5) we have $|\alpha|^2|\beta|^2 = |\alpha|^2(1+|\gamma|^2)$. It follows that $|\alpha| = |\gamma|$, or

$$\gamma = e^{i\phi}\alpha\,. \tag{6}$$

Suppose $\alpha \neq 0$. Then from (4) and (5) we can determine β, obtaining

$$\beta = \frac{e^{-i\theta}e^{i\phi}\alpha\sqrt{1+|\alpha|^2}}{\overline{\alpha}}\,. \tag{7}$$

On the other hand, if $\alpha = 0$ then also $\gamma = 0$, hence from (5) and (3) we conclude that $n = \beta$ is a constant of absolute value 1. (Thus (7) remains in some sense valid when $\alpha = 0$.) Equations (3) and (7) give

$$n = \alpha m + \frac{e^{-i\theta}e^{i\phi}\alpha\sqrt{1+|\alpha|^2}}{\overline{\alpha}} \qquad \text{and} \qquad q = e^{i\theta}m\sqrt{1+|\alpha|^2} + e^{i\phi}\alpha\,,$$

and then

$$m = \frac{q - e^{i\phi}\alpha}{e^{i\theta}\sqrt{1+|\alpha|^2}}.$$

Thus

$$P = m^2 p = \frac{(q-e^{i\phi}\alpha)^2}{e^{2i\theta}(1+|\alpha|^2)}p = \frac{(e^{-i\phi}q-\alpha)^2}{1+|\alpha|^2}e^{2i(\phi-\theta)}p\,,$$

and

$$Q = \frac{n}{m} = \alpha + \frac{e^{i\phi}\alpha(1+|\alpha|^2)}{\overline{\alpha}(q-e^{i\phi}\alpha)} = \frac{\alpha}{\overline{\alpha}}\left(\frac{\overline{\alpha}q + e^{i\phi}}{q-e^{i\phi}\alpha}\right) = -\frac{\alpha}{\overline{\alpha}}\left(\frac{\overline{\alpha}e^{-i\phi}q+1}{\alpha - e^{-i\phi}q}\right),$$

or

$$(P,Q) = R_q(\pi)I(\alpha)R_p(2\phi-2\theta)R_q(-\phi)(p,q) \tag{8}$$

in terms of the group. Equation (8) also holds for $\alpha = 0$.

We still have the earlier case to analyze when m is constant. Equation (1) then becomes

$$1 + |q|^2 = |m|^2(1+|Q|^2), \tag{9}$$

and taking the Laplacian of both sides results in $|q'| = |m||Q'|$. Thus $q = e^{i\phi}mQ+\mu$, which inserted back into (9) gives

$$1 + |m|^2|Q|^2 + |\mu|^2 + 2\,\mathrm{Re}\{e^{i\phi}m\overline{\mu}Q\} = |m|^2(1+|Q|^2)\,. \tag{10}$$

Hence $\mathrm{Re}\{e^{i\phi}m\overline{\mu}Q\}$ is constant, so Q is constant unless $\mu = 0$. If $\mu = 0$, then (10) implies that $|m| = 1$, and the pair (P,Q) is a rotation of (p,q). More precisely,

$$(P,Q) = R_q(\theta+\phi)R_p(2\theta)(p,q)\,, \qquad \text{where } m = e^{i\theta}\,.$$

If Q is constant then, in light of (9), q is also constant. In this case the minimal surfaces associated with (p,q) and (P,Q) are planar, and up to a rotation of $\mathbb{R}^3$, the Weierstrass–Enneper lifts $\widetilde{f}$ and $\widetilde{F}$ are analytic with equal Schwarzians.

We describe the effects of the action of the group on the minimal surface associated with (p,q). First note that if $\xi = re^{i\theta}$ then

$$I(\xi) = R_q(\theta)R_p(2\theta)I(r)R_q(-\theta)\,.$$

Thus up to rotations of p and q we may assume that the parameter defining the transformation I is real and nonnegative. Suppose, then, that

$$P = \frac{(q-r)^2}{1+r^2}\,p\,, \quad Q = \frac{rq+1}{r-q}, \qquad r > 0\,,$$

and let (u,v,w), (U,V,W) be the coordinates in $\mathbb{R}^3$ associated with (p,q) and (P,Q), through the Weierstrass–Enneper equations. A simple calculation shows that $P(1+Q^2) = p(1+q^2)$, hence $U = u$. We also find that

$$P(1-Q^2) = -\left(\frac{1-r^2}{1+r^2}\right)p(1-q^2) - \left(\frac{4r}{1+r^2}\right)pq,$$

and

$$PQ = \left(\frac{r}{1+r^2}\right)p(1-q^2) - \left(\frac{1-r^2}{1+r^2}\right)pq.$$

Hence

$$V = -\left(\frac{1-r^2}{1+r^2}\right)v - \left(\frac{2r}{1+r^2}\right)w \qquad \text{and} \qquad W = \left(\frac{2r}{1+r^2}\right)v - \left(\frac{1-r^2}{1+r^2}\right)w.$$

The map $(v,w) \mapsto (V,W)$ is a 2-dimensional rotation, and so $(u,v,w) \mapsto (U,V,W)$ is a rotation of $\mathbb{R}^3$.

We now examine the rotations $R_p(\theta)$ and $R_q(\phi)$. We introduce the local harmonic conjugates to u, v, and w in the form:

$$u + iu^* = \int p(1+q^2)\,dz, \quad v^* + iv = \int p(1-q^2), \quad w^* + iw = 2\int pq\,dz\,.$$

A rotation of p by $e^{i\theta}$ gives

$$U = u\cos\theta - u^*\sin\theta\,, \qquad V = v\cos\theta + v^*\sin\theta\,, \qquad W = w\cos\theta + w^*\sin\theta\,.$$

Since

$$h = \int p\,dz \qquad \text{and} \qquad g = \int pq^2\,dz\,,$$

it follows that

$$h = \tfrac{1}{2}((u+v^*) + i(u^*+v)) \qquad \text{and} \qquad g = \tfrac{1}{2}((u-v^*) + i(u^*-v)).$$

A rotation of q by $e^{i\phi}$ affects g but not h, and we have

$$U = \mathrm{Re}\{h + e^{2i\phi}g\} = u\cos^2\phi - u^*\sin\phi\cos\phi + v\sin\phi\cos\phi + v^*\sin^2\phi\,,$$
$$V = \mathrm{Im}\{h - e^{2i\phi}g\} = -u\sin\phi\cos\phi + u^*\sin^2\phi + v\cos^2\phi + v^*\sin\phi\cos\phi\,,$$
$$W = \mathrm{Im}\{e^{i\phi}(w^* + iw)\} = w\cos\phi + w^*\sin\phi\,.$$

Since each component U, V, W also has a local harmonic conjugate, one can view the elements of $\mathcal{G}$ as linear maps from (u, v, w, u^*, v^*, w^*) to (U, V, W, U^*, V^*, W^*), followed by projection onto the first three components. In this identification, it is not difficult to see that the action in 6 dimensions is orthogonal, showing, for example, that a linear change in the components u, v will not arise from an element in $\mathcal{G}$ unless it is a rotation. On the other hand, one can verify directly that a rotation $(u, v) \mapsto (u\cos\theta - v\sin\theta, u\sin\theta + v\cos\theta)$ is the result of applying $R_p(-\theta)$ and $R_q(\theta)$. Since for a real parameter, the transformation I is a rotation in the (v, w)-plane, it follows that with elements in $\mathcal{G}$ we can recover all rotations of 3-space.

We mention that if (u, v, w) is a conformal parametrization of a minimal surface by harmonic functions then their harmonic conjugates u^*, v^*, w^* parametrize conformally a minimal surface known in the literature as the *adjoint surface*. The Cauchy–Riemann equations imply that a minimal surface and its adjoint surface are isometric. An interesting observation is that a minimal surface and its adjoint have the same tangent plane at corresponding points. We do not have a clear geometric description of the action of the group $\mathcal{G}$ in general when both the surface and its adjoint are involved in the equations.

§4. Geometric derivation of the Schwarzian formula.

The link between harmonic and conformal mappings allows us to introduce the Schwarzian derivative as developed in [9] for conformal changes of metrics and conformal mappings of Riemannian manifolds. Let $(M, \mathbf{g})$ be an n-dimensional Riemannian manifold, $n \geq 2$. A *conformal metric* $\hat{\mathbf{g}}$ is a multiple of $\mathbf{g}$ by a positive smooth function, called the conformal factor. Write this in the form $\hat{\mathbf{g}} = e^{2\sigma}\mathbf{g}$. Two conformal metrics are *homothetic* if the conformal factor is a positive constant. The *Schwarzian tensor* of σ (or of $\hat{\mathbf{g}}$) is defined as

$$B_{\mathbf{g}}(\sigma) = \mathrm{Hess}(\sigma) - d\sigma \otimes d\sigma - \tfrac{1}{n}(\Delta\sigma - \|\mathrm{grad}\ \sigma\|^2)\mathbf{g}\,, \tag{11}$$

where the metric-dependent quantities (the Hessian, Laplacian and gradient) are computed with respect to $\mathbf{g}$. The Schwarzian tensor $B_{\mathbf{g}}(\sigma)$ is a symmetric, traceless 2-tensor.

If $f : (M, \mathbf{g}) \to (M', \mathbf{g}')$ is a conformal local diffeomorphism between Riemannian manifolds, with $f^*\mathbf{g}' = e^{2\sigma}\mathbf{g}$, then its *Schwarzian derivative* is defined as the tensor $\mathcal{S}_{\mathbf{g}}f = B_{\mathbf{g}}(\sigma)$. A *Möbius transformation* is a conformal diffeomorphism with $\mathcal{S}_{\mathbf{g}}f = 0$. This includes the usual Möbius transformations in two dimensions as a special case.

We now make the following definition, which applies when the complex dilatation is (locally) the square of an analytic function.

Definition. *Let f be a harmonic mapping of a domain Ω, let $\widetilde{f} : \Omega \to \mathbb{R}^3$ be its Weierstrass–Enneper lift to a minimal surface, and let $\rho_f|dz|$ be the corresponding conformal metric. Then the Schwarzian derivative of f is defined by*

$$Sf = \mathcal{S}_{\mathbf{g}_0}\tilde{f} = B_{\mathbf{g}_0}(\log \rho_f)\,,$$

where $\mathbf{g}_0$ is the Euclidean metric.

In [9] it was assumed that the source and target manifolds have the same dimension, but it is clear that the definition of the Schwarzian applies in the present situation.

In two dimensions and for conformal mappings in the Euclidean metric, it is helpful to represent the tensor $B_{\mathbf{g_0}}(\log \rho_f)$ in standard coordinates as a 2×2 symmetric, traceless matrix. If $f(z)$ is analytic, then $\rho_f(z) = |f'(z)|$ and one can show directly from (1) that

$$\mathcal{S}_{\mathbf{g_0}} f = \begin{pmatrix} \mathrm{Re}\{S(f)\} & -\mathrm{Im}\{S(f)\} \\ -\mathrm{Im}\{S(f)\} & -\mathrm{Re}\{S(f)\} \end{pmatrix},$$

where $S(f)$ now denotes the classical Schwarzian derivative,

$$S(f) = (f''/f')' - \tfrac{1}{2}(f''/f')^2\,.$$

In general, the Schwarzian tensor of a conformal metric $e^{2\sigma}|dz|^2$ relative to the Euclidean metric is given by a matrix of the form

$$\begin{pmatrix} a & -b \\ -b & -a \end{pmatrix},$$

where

$$a + ib = 2(\sigma_{zz} - \sigma_z^2)\,.$$

This natural identification of the tensor with a complex number provides the formula

$$S(f) = 2\big\{(\log \rho)_{zz} - ((\log \rho)_z)^2\big\}$$

for the Schwarzian derivative of a harmonic mapping, which we have already explored in Section 2.

5. Proof of Theorem 3.

We now give the proof of Theorem 3. Suppose that $Sf = SF$; that is,

$$B_{\mathbf{g_0}}(\log \rho_f) = B_{\mathbf{g_0}}(\log \rho_F)\,, \tag{12}$$

where $\mathbf{g_0}$ is the Euclidean metric. We approach the problem of solutions of this equation via linearizing, somewhat as is done for the Schwarzian of analytic functions in the classical setting. For background and the general theory supporting the following discussion we refer to [9].

There is an addition formula for the tensor B given in [9]. In its general form it says that

$$B_{\mathbf{g_0}}(\varphi + \sigma) = B_{\mathbf{g_0}}(\varphi) + B_{\mathbf{g}}(\sigma)\,,$$

where $\mathbf{g} = e^{2\varphi}\mathbf{g_0}$. (This is a generalization of the chain rule for the Schwarzian of the composition of two analytic functions.) We apply this when $\mathbf{g_0}$ is the Euclidean metric, when the conformal metric is

$$\mathbf{g} = \rho_f^2\, \mathbf{g_0}\,,$$

and when

$$\varphi = \log \rho_f \quad \text{and} \quad \sigma = \log \rho_F - \log \rho_f.$$

From (12) we then obtain

$$B_{\mathbf{g}}(\log \rho_F - \log \rho_f) = 0. \tag{13}$$

The change of variable

$$u = \frac{\rho_f}{\rho_F}$$

converts (13) to the *linear* equation

$$\operatorname{Hess} u = \tfrac{1}{2}(\Delta u)\, \mathbf{g}\,, \tag{14}$$

where the Hessian and Laplacian are computed with respect to the metric $\mathbf{g}$.

The existence of a nonconstant solution to (14) has strong consequences for the metric. In the form we need, they derive from the following lemma. A more general version of this result is in [9], but we will give the proof of this case here to illustrate the methods.

Lemma 1. *Let u be a solution to equation* (14)*. Then in a neighborhood of a point where* grad $u \neq 0$,

(a) *the integral curves to* grad u *are geodesics; and*
(b) *the orthogonal trajectories have constant geodesic curvature.*

Proof of lemma. Let ∇ denote covariant differentiation in the metric $\mathbf{g}$, and let T and N be orthonormal vector fields on a neighborhood of a point where grad $u \neq 0$, with T in the direction of grad u. We note some simple, general identities for the covariant derivatives of T and N:

$$\nabla_T T = \alpha N, \quad \nabla_N N = \beta T, \quad \nabla_T N = -\alpha T, \quad \nabla_N T = -\beta N,$$

where α and β are functions on the neighborhood. These follow from differentiating the orthonormality conditions in the T and N directions. Under the assumptions of the lemma, we claim that actually $\nabla_T T = 0$ on the neighborhood; *i.e.*, that $\alpha = 0$. This will prove part (a), since $\nabla_T T = \kappa N$ along a T-curve (an integral curve for grad u), where κ is the geodesic curvature. To deduce this we compute

$$\nabla_T T = \nabla_T\left(\frac{\text{grad } u}{\|\text{grad } u\|}\right) = \frac{1}{\|\text{grad } u\|}\,\nabla_T(\text{grad } u) + T\left(\frac{1}{\|\text{grad } u\|}\right)\text{grad } u\,,$$

hence

$$\alpha = \mathbf{g}(\nabla_T T, N) = \frac{1}{\|\text{grad } u\|}\,\mathbf{g}(\nabla_T(\text{grad } u), N) = \frac{1}{\|\text{grad } u\|}\,\text{Hess } u(T,N) = 0\,,$$

by (14). An additional consequence is that $\nabla_T N = 0$.

We turn now to part (b). Since $\nabla_N N = \beta T$, we need to show that β is constant in the N-direction. As before we compute

$$\nabla_N T = \nabla_N\left(\frac{\text{grad } u}{\|\text{grad } u\|}\right) = \frac{1}{\|\text{grad } u\|}\nabla_N(\text{grad } u) + N\left(\frac{1}{\|\text{grad } u\|}\right)\text{grad } u\,,$$

so that

$$\beta = -\mathbf{g}(\nabla_N T, N) = -\frac{1}{\|\text{grad } u\|}\,\text{Hess } u(N,N) = -\frac{1}{2\|\text{grad } u\|}\,\Delta u\,.$$

Now,

$$N(\|\text{grad } u\|^2) = 2\,\mathbf{g}(\nabla_N(\text{grad } u), \text{grad } u) = 2\,\text{Hess } u(\text{grad } u, N) = 0\,,$$

and hence $\|\text{grad } u\|$ is constant in the N-direction. Furthermore,

$$T(\|\text{grad } u\|^2) = 2\,\mathbf{g}(\nabla_T(\text{grad } u), \text{grad } u) = 2\,\text{Hess } u(\text{grad } u, T) = \|\text{grad } u\|\Delta u\,,$$

so we have the expression

$$\Delta u = \frac{1}{\|\text{grad } u\|}T(\|\text{grad } u\|)\,.$$

We will derive $N(\Delta u) = 0$, thus showing that Δu is constant in the N-direction and completing the proof. For this, note first that

$$N(\Delta u) = N\left(\frac{1}{\|\text{grad } u\|}\right)T(\|\text{grad } u\|^2) + \frac{1}{\|\text{grad } u\|} NT(\|\text{grad } u\|^2)$$
$$= \frac{1}{\|\text{grad } u\|} NT(\|\text{grad } u\|^2)\,,$$

because $\|\text{grad } u\|$ is constant in the N-direction. But now, using $NT - TN = \nabla_N T - \nabla_T N = \nabla_N T = -\beta N$, we obtain

$$NT(\|\text{grad } u\|^2) = (TN + \nabla_N T)(\|\text{grad } u\|^2) = (TN - \beta N)(\|\text{grad } u\|^2) = 0\,,$$

which completes the proof of Lemma 1.

Lemma 1 shows that, unless u is constant, there exist a neighborhood and a network of T and N curves, briefly a (T, N)-rectangle, on which $\mathbf{g}$ is given by the *warped product metric*

$$dr^2 + \ell(r)^2 d\theta^2\,,$$

where $\ell(r)$ is the length along an integral curve of N as a function of the distance r along integral curves of T. The curvature of this metric is simply

$$K(r) = -\frac{\ell''(r)}{\ell(r)}\,,$$

a function only of r.

It now follows that there is a conformal model of this metric, $\lambda^2|d\zeta|^2$ — *not* (necessarily) the original conformal metric $\mathbf{g} = \rho_f^2|dz|^2$ — in which the conformal factor λ is *radial*; *i.e.*, a function of $|\zeta|$. In this model the T-curves correspond to (Euclidean) polar rays and the N-curves to Euclidean circles $|\zeta| =$ const. To establish the isometry (locally) between $dr^2 + \ell(r)^2 d\theta^2$ and $\lambda^2(|\zeta|)|d\zeta|^2$, it suffices to ensure that the curvatures of the two metrics are equal. This condition is the existence of a local solution of the equation

$$-\lambda^{-2}\left(\left(\frac{\lambda'}{\lambda}\right)' + \frac{1}{|\zeta|}\frac{\lambda'}{\lambda}\right) = K(r)\,,$$

where

$$r = \int_{|\zeta_0|}^{|\zeta|} \lambda(s)\,ds\,;$$

different choices of the initial value $|\zeta_0|$ correspond to homothetic changes in the metric $\lambda|d\zeta|$.

To summarize, we have an isometry $z = \jmath(\zeta)$ of a region of the ζ-plane with the radial metric $\lambda(|\zeta|)^2|d\zeta|^2$ to a (T, N)-rectangle in the z-plane with the metric

$\rho_f^2(z)|dz|^2$. In particular, as $\jmath$ is conformal in the Euclidean metric it is an analytic function, and so

$$\lambda(|\zeta|) = \rho_f(\jmath(\zeta))|\jmath'(\zeta)|\,; \quad i.e., \quad \log\lambda(|\zeta|) = \log\rho_f(\jmath(\zeta)) + \log|\jmath'(\zeta)|.$$

Likewise we put

$$\log\mu(\zeta) = \log\rho_F(\jmath(\zeta)) + \log|\jmath'(\zeta)|\,, \quad \text{so that} \quad \mu(\zeta)^2|d\zeta|^2 = \jmath^*(\rho_F(z)^2|dz|^2).$$

Since $\jmath$ is an isometry between $\mathbf{g_1} = \lambda(|\zeta|)^2|d\zeta|^2$ and $\mathbf{g} = \rho_f^2(z)|dz|^2$, it follows from (13) that

$$B_{\mathbf{g_1}}(\log(\rho_F\circ\jmath) - \log(\rho_f\circ\jmath)) = 0\,, \quad \text{or} \quad B_{\mathbf{g_1}}(\log(\rho_F\circ\jmath) + \log|\jmath'| - \log\lambda) = 0\,;$$

that is,

$$B_{\mathbf{g_1}}(\log\mu - \log\lambda) = 0\,.$$

The addition formula for B implies that

$$B_{\mathbf{g_0}}(\log\lambda) = B_{\mathbf{g_0}}(\log\mu)\,,$$

where $\mathbf{g_0}$ is the Euclidean metric. As before, $v = \lambda/\mu$ is a solution of

$$\text{Hess}\, v = \tfrac{1}{2}(\Delta v)\mathbf{g_1}\,, \tag{15}$$

where the Hessian and Laplacian are computed with respect to $\mathbf{g_1}$.

Now, because λ is radial the metric $\mathbf{g_1} = \lambda(|\zeta|)^2|d\zeta|^2$ is a warped product metric; if $R = \int \lambda(|\zeta|)\,d|\zeta|$, then it is just

$$dR^2 + ((R')^2|\zeta|^2)d\theta^2.$$

We are now in a position to apply Theorem 5.3 in [9], which gives a precise description of the form of solutions to (15) in a warped product metric. According to that theorem there are two cases:

- (i) All solutions are functions only of the variable in the first term in the warped product, in this case $|\zeta|$.
- (ii) The metric $\mathbf{g_1}$ has constant curvature, in which case some additional solutions may occur.

We will deal with (ii) at the end of the section; as we have seen before, constant curvature means zero curvature in the context of harmonic maps, so this case will be easy to analyze.

Suppose, then, that v is radial. Since λ is radial, so therefore is μ. It follows from the relation $\rho = |h'|+|g'|$ that, except at isolated points, the conformal factors ρ_f and ρ_F are sums of squares of two analytic functions, so the same must be true of λ and μ (incorporating the factor $|\jmath'|$). Determining all such possibilities can be phrased as a problem in complex analysis. We put it this way:

Lemma 2. *Let $\phi(\zeta)$ and $\psi(\zeta)$ be analytic in a domain Ω, and suppose that $|\phi(\zeta)|^2 + |\psi(\zeta)|^2$ depends only on $|\zeta|$. Then there exist a unitary matrix U, complex numbers A and B, and real numbers α and β such that*

$$\begin{pmatrix} \phi(\zeta) \\ \psi(\zeta) \end{pmatrix} = U \begin{pmatrix} A\zeta^\alpha \\ B\zeta^\beta \end{pmatrix}.$$

We defer the proof to the end of this section. To apply the lemma, we have to compute the Schwarzian tensor of a conformal metric $\lambda|d\zeta|$ of the form

$$\lambda(\zeta) = a|\zeta|^{2\alpha} + b|\zeta|^{2\beta},$$

with $a = |A|^2$, $b = |B|^2$ as in the lemma. We find this to be

$$-\zeta^{-2}\,\frac{\alpha(\alpha+1)a^2|\zeta|^{4\alpha} + \beta(\beta+1)b^2|\zeta|^{4\beta} + (4\alpha\beta - \alpha(\alpha-1) - \beta(\beta-1))ab|\zeta|^{2(\alpha+\beta)}}{(a|\zeta|^{2\alpha} + b|\zeta|^{2\beta})^2}.$$

A second metric with conformal factor $\mu(\zeta) = a'|\zeta|^{2\alpha'} + b'|\zeta|^{2\beta'}$ will have the same Schwarzian if and only if the exponents are the same, say $\alpha' = \alpha$, $\beta' = \beta$, and $b'/a' = b/a$. In other words, the conformal metrics $\lambda|d\zeta|$ and $\mu|d\zeta|$ are homothetic. Returning to the earlier situation, we see that this implies $\rho_f|dz|$ and $\rho_F|dz|$ are homothetic, as we were to show. If the metrics are homothetic, their curvatures are equal and the Schwarzians of f and F are equal.

It remains to discuss the case (ii), where the metric $\mathbf{g_1} = \lambda^2|d\zeta|^2$ has constant curvature. But then so does $\rho_f^2|dz|^2$, since they are isometric. Furthermore, because the argument is symmetric in f and F, if we are not in the case considered already it must also be that $\rho_F^2|dz|^2$ has constant curvature. Hence the minimal surfaces $\widetilde{f}(\Omega)$ and $\widetilde{F}(\Omega)$ have constant curvature, which implies they are planar. Therefore, as in the analysis of vanishing Schwarzians, the conformal factors are of the form

$$\rho_f = (1 + |q_0|^2)|p| \quad \text{and} \quad \rho_F = (1 + |Q_0|^2)|P|,$$

where q_0 and Q_0 are constants. If $h' = p$, $H' = P$, then we have $S(h) = S(H)$ (analytic Schwarzian), and hence H is an analytic Möbius transformation of h. Conversely, if (p, q) and (P, Q) are related in this way, then the curvatures are zero and $S(f) = S(F)$. This completes the proof of Theorem 3, subject to a proof of Lemma 2.

Proof of Lemma 2. The lemma is easy to prove through straightforward use of power series if one knows that ϕ and ψ are defined in a neighborhood of the origin; in this case α and β are integers. Otherwise, choose a rectangle

$$R = \{w = s + it \;:\; s_1 < s < s_2\,,\; t_1 < t < t_2\}$$

that is mapped conformally onto a polar rectangle in Ω by $\zeta = e^w$. Abusing the notation slightly, we then have two analytic functions $\phi(w)$ and $\psi(w)$ on R

with the property that $|\phi(s+it)|^2 + |\psi(s+it)|^2$ is independent of t. We first show that ϕ and ψ can be continued analytically to the entire strip $s_1 < s < s_2$, $-\infty < t < \infty$. In this strip, $|\phi|^2 + |\psi|^2$ will remain a function only of s, since $(\partial/\partial t)(|\phi(s+it)|^2 + |\psi(s+it)|^2)$ already vanishes on an open set.

Let $w_0 = s_0 + it_0$, $w_1 = s_0 + i(t_0 + \tau)$ be two points in R with the same real part. We consider the power series expansions of ϕ and ψ near w_0 and w_1:

$$\phi(w) = \sum_{n=0}^{\infty} a_n (w - w_0)^n \,, \qquad \psi(w) = \sum_{n=0}^{\infty} b_n (w - w_0)^n \,,$$

$$\phi(w) = \sum_{n=0}^{\infty} A_n (w - w_1)^n \,, \qquad \psi(w) = \sum_{n=0}^{\infty} B_n (w - w_1)^n \,.$$

The hypothesis $|\phi(w)|^2 + |\psi(w)|^2 = |\phi(w+i\tau)|^2 + |\psi(w+i\tau)|^2$ implies that

$$\sum_{n,m=0}^{\infty} (a_n \overline{a_m} + b_n \overline{b_m})(w - w_0)^n (\overline{w} - \overline{w_0})^m = \sum_{n,m=0}^{\infty} (A_n \overline{A_m} + B_n \overline{B_m})(w - w_0)^n (\overline{w} - \overline{w_0})^m$$

for all w near w_0. Hence

$$a_n \overline{a_m} + b_n \overline{b_m} = A_n \overline{A_m} + B_n \overline{B_m} \tag{16}$$

for all n and m. In particular $|a_n|^2 + |b_n|^2 = |A_n|^2 + |B_n|^2$, so that

$$\max\{|A_n|, |B_n|\} \leq |a_n| + b_n| \qquad \text{and} \qquad \max\{|a_n|, |b_n|\} \leq |A_n| + |B_n|.$$

We conclude that the radii of convergence of power series for ϕ and ψ, respectively, do not depend on t, and this ensures that ϕ and ψ can be continued analytically to the entire strip.

Next, the relation in (16) states that the set of vectors $\{(a_n, b_n)\}$ in $\mathbb{C}^2$ has pairwise the same inner products as does the set of vectors $\{(A_n, B_n)\}$. Hence there is a unitary matrix V with

$$\begin{pmatrix} A_n \\ B_n \end{pmatrix} = V \begin{pmatrix} a_n \\ b_n \end{pmatrix} \,,$$

and from this we can write

$$\begin{pmatrix} \phi(w+i\tau) \\ \psi(w+i\tau) \end{pmatrix} = V \begin{pmatrix} \phi(w) \\ \psi(w) \end{pmatrix}$$

for w in a neighborhood of w_0.

Now since V is unitary, it can be diagonalized via conjugation by another unitary matrix:

$$\begin{pmatrix} e^{2\pi i\theta_1} & 0 \\ 0 & e^{2\pi i\theta_2} \end{pmatrix} = WVW^* \,, \qquad W \text{ unitary.}$$

Define analytic functions $\Phi(w)$ and $\Psi(w)$ in the strip by

$$\begin{pmatrix} \Phi(w) \\ \Psi(w) \end{pmatrix} = W \begin{pmatrix} \phi(w) \\ \psi(w) \end{pmatrix} .$$

Then for w in a neighborhood of w_0 we have

$$\Phi(w+i\tau) = e^{2\pi i\theta_1}\Phi(w) \qquad \text{and} \qquad \Psi(w+i\tau) = e^{2\pi i\theta_2}\Psi(w) .$$

Here θ_1 and θ_2 depend on τ. If we fix $\tau = 2\pi$ and multiply Φ and Ψ by $e^{-\theta_1 w}$ and $e^{-\theta_2 w}$, respectively, we obtain analytic functions that are $2\pi i$ - periodic in the strip. We use this fact to write

$$\Phi(w) = e^{-\theta_1 w} \sum_{n=-\infty}^{\infty} \alpha_n e^{2\pi n w} , \qquad \Psi(w) = e^{-\theta_2 w} \sum_{n=-\infty}^{\infty} \beta_n e^{2\pi n w} ,$$

so that

$$\begin{aligned} |\Phi(s+it)|^2 + |\Psi(s+it)|^2 = e^{2\theta_1 s} &\sum_{n,m} \alpha_n \overline{\alpha_m} e^{2\pi(n+m)s} e^{2\pi i(n-m)t} \\ + e^{2\theta_2 s} &\sum_{n,m} \beta_n \overline{\beta_m} e^{2\pi(n+m)s} e^{2\pi i(n-m)t} . \end{aligned} \tag{17}$$

Since $|\Phi|^2 + |\Psi|^2 = |\phi|^2 + |\psi|^2$ (because the pairs of functions are related by a unitary transformation) the sum in (17) is still a function only of s. There are now two possibilities:

(i) $\theta_1 = \theta_2$ and $\alpha_n\overline{\alpha_m} + \beta_n\overline{\beta_m} = 0$ for all $n \neq m$.
(ii) $\theta_1 \neq \theta_2$ and $\alpha_n\overline{\alpha_m} = \beta_n\overline{\beta_m} = 0$ for all $n \neq m$.

In case (ii) each sum in (17) consists of just one term. In case (i) there are at most two indices for which the pair (α_n, β_n) is not $(0,0)$, and they satisfy the relation in (i). To see this, suppose for example that $\alpha_1 \neq 0$. Then from $\alpha_1\overline{\alpha_n} + \beta_1\overline{\beta_n} = 0$ all remaining pairs (α_n, β_n) must satisfy the single linear relation $\alpha_n = c\beta_n$, $c = \overline{(\beta_1/\alpha_1)}$. For two pairs (α_n, β_n), (α_m, β_m) we would then have

$$0 = \alpha_n\overline{\alpha_m} + \beta_n\overline{\beta_m} = (1+|c|^2)\beta_n\overline{\beta_m} .$$

This implies that there cannot be two pairs with nonzero second component, and if the second component vanishes then so does the first.

In either case, unwinding the definitions back to ϕ and ψ leads directly to the assertion of Lemma 2. This proves the lemma and completes the proof of Theorem 3.

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